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Global bounds for the Lyapunov exponent and the integrated density of states of random Schrödinger operators in one dimension

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Abstract. In this paper we prove an upper bound for the Lyapunov exponent $\gamma(E)$ and a two-sided bound for the integrated density of states $N(E)$ at an arbitrary energy $E > 0$ of random Schrödinger operators in one dimension. These Schrödinger operators are given by potentials of identical shape centred at every lattice site but with non-overlapping supports and with randomly varying coupling constants. Both types of bounds only involve scattering data for the single-site potential. They show, in particular, that both $\gamma(E)$ and $N(E) - \sqrt{E}/\pi$ decay at infinity at least like $1/\sqrt{E}$. As an example we consider the random Kronig–Penney model.

1. Introduction

In this paper we will consider random Schrödinger operators $H(\omega)$ in $L^2(\mathbb{R})$ of the form

$$H(\omega) = H_0 + V_\omega \qquad H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \qquad V_\omega = \sum_{j\in\mathbb{Z}} \alpha_j(\omega) f(\cdot - j) \tag{1}$$

where $\{\alpha_j(\omega)\}_{j\in\mathbb{Z}}$ is a sequence of independent, identically distributed (IID) variables on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having a common distribution measure κ (i.e. $\mathbb{P}\{\alpha_j \in \Delta\} = \kappa(\Delta)$ for any Borel set $\Delta \subset \mathbb{R}$). In what follows we always suppose that κ is supported on a compact interval and the single-site potential f is integrable with support in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, the random variables are assumed to form a stationary, metrically transitive random field, i.e. there are measure-preserving ergodic transformations $\{T_j\}_{j\in\mathbb{Z}}$ such that $\alpha_j(T_k\omega) = \alpha_{j-k}(\omega)$ for all $\omega \in \Omega$. The spectral properties of the operator (1) were studied in detail in [7, 9, 11, 16, 24]. The results are most complete for the case when f is the point interaction (see [1]).

The integrated density of states $N(E)$ and the Lyapunov exponent $\gamma(E)$ are important quantities associated with operators of the form (1) (see, e.g., [4]). In particular, according to the Ishii–Pastur–Kotani theorem [15] the set $\{E : \gamma(E) = 0\}$ is the essential support of the absolute continuous part of the spectral measure for $H(\omega)$.

The main idea of our approach is to approximate the operator (1) by means of the sequence

$$H^{(n)}(\omega) = H_0 + \sum_{j=-n}^{n} \alpha_j(\omega) f(\cdot - j)$$

with unchanged H_0, which converges to $H(\omega)$ in the strong resolvent sense. This differs from the usual approach where one puts the whole system in a box, which then tends to infinity (see, e.g., [4]). In [12] (see also [13]) we used this approximation to invoke scattering theory for the study of the spectral properties of the limiting operator (1). Some other applications of scattering theory to the study of spectral properties of such types of Schrödinger operators in one dimension can be found in [11, 22].

One of the important ingredients of our approach developed in [12] is the Lifshitz–Krein spectral shift function. The spectral shift function naturally replaces the eigenvalue counting function usually used to construct the density of states for the operator (1). The celebrated Birman–Krein theorem (see, e.g., [3]) relates the spectral shift function to scattering theory. In fact, up to a factor of $-\pi^{-1}$ it may be identified with the scattering phase for the pair $(H^{(n)}(\omega), H_0)$, i.e. $\xi^{(n)}(E;\omega) = -\pi^{-1}\delta^{(n)}(E;\omega)$ when $E > 0$,

$$\delta^{(n)}(E;\omega) = \frac{1}{2\mathrm{i}} \log\det S^{(n)}(E;\omega) = \frac{1}{2\mathrm{i}} \log\det \begin{pmatrix} T_\omega^{(n)}(E) & R_\omega^{(n)}(E) \\ L_\omega^{(n)}(E) & T_\omega^{(n)}(E) \end{pmatrix}.$$

Here $|T^{(n)}(E)|^2$ and $|R^{(n)}(E)|^2 = |L^{(n)}(E)|^2$ have the meaning of transmission and reflection coefficients, respectively, such that $|T^{(n)}(E)|^2 + |R^{(n)}(E)|^2 = 1$. For $E < 0$ the quantity $\xi^{(n)}(E;\omega)$ equals minus the counting function for $H^{(n)}(\omega)$.

In particular, in [12] we proved the almost certain existence of the limit

$$\xi(E) = \lim_{n\to\infty} \frac{\xi^{(n)}(E;\omega)}{2n+1} \tag{2}$$

which we called the spectral shift density. Also we proved the equality $\xi(E) = N_0(E) - N(E)$, where $N(E)$ and $N_0(E) = \pi^{-1}[\max(0, E)]^{1/2}$ are the integrated density of states of the Hamiltonians $H(\omega)$ and H_0, respectively. This result also extends to higher dimension in the continuous [14] and discrete [5] cases. Also we showed that almost certainly the Lyapunov exponent $\gamma(E)$ at energy $E > 0$ is given as

$$\gamma(E) = -\lim_{n\to\infty} \frac{\log|T^{(n)}(E;\omega)|}{2n+1} \tag{3}$$

where $T^{(n)}(E,\omega)$ is the transmission amplitude for the pair of Hamiltonians $(H^{(n)}(\omega), H_0)$ at energy E. We recall that $\gamma(E)$ is defined as the upper Lyapunov exponent for the fundamental matrix at energy E of the Schrödinger operator $H(\omega)$. The connection between the Lyapunov exponent and the transmission coefficient $|T_\omega^{(n)}(E)|$ was recognized long ago [17, 18]. A complete proof has appeared in [12].

We note that the theory of the spectral shift function was also recently used to show that the integrated density of states is independent of the choice of boundary conditions [19] on the sides of a large box, in which the system is put.

The conditions on the random variables α_j and the single-site potential f stated above are slightly weaker than those in [12]. However, the results of [12] which will be used below also remain valid in this more general case.

The aim of the present paper is to prove *global* bounds for the Lyapunov exponent and the integrated density of states, i.e. bounds which hold for all $E > 0$ and describe the correct asymptotic behaviour in the limit $E \to \infty$. These results are formulated as theorems 1 and

2 below. To the best of our knowledge the first article to look for the asymptotic behaviour of $\gamma(E)$ and $N(E)$ in the limit $E \to \infty$ was [2]. The best known estimate for the integrated density of states is due to Kirsch and Martinelli [10, corollary 3.1]. This bound, however, does not reproduce the correct asymptotic behaviour of $N(E)$ in the large-energy limit. Another estimate, which is due to Pastur and Figotin (see [20, section 5.11.B]), is valid for an $\mathbb{R}$-metrically transitive random field. Since our potential $V_\omega(x)$ is a $\mathbb{Z}$-metrically transitive field this estimate does not apply directly to the present situation. Our two-sided estimate leads to the bound (23) below which is very close to that of Pastur and Figotin.

In what follows C will denote a finite positive generic constant varying with the context, but which depends only on f and κ.

2. The Lyapunov exponent

We recall that the scattering matrix $S(E)$ for a pair of Hamiltonians (H, H_0) on $L^2(\mathbb{R})$ at fixed energy $E \geqslant 0$ is a 2×2 unitary matrix

$$S(E) = \begin{pmatrix} T(E) & R(E) \\ L(E) & T(E) \end{pmatrix} \tag{4}$$

where $L(E)$ and $R(E)$ denote the left and right reflection amplitudes, respectively. The transmission amplitude $T(E)$ can vanish only for $E = 0$ (see [6, 8]). To any S-matrix (4) we associate the unimodular matrix

$$\Lambda(E) = \begin{pmatrix} \dfrac{1}{T(E)} & -\dfrac{R(E)}{T(E)} \\ \dfrac{L(E)}{T(E)} & \dfrac{1}{\overline{T(E)}} \end{pmatrix}.$$

Let $T_\alpha(E)$, $R_\alpha(E)$ and $L_\alpha(E)$ be the elements of the S-matrix at energy E for the pair of operators $(H_0 + \alpha f, H_0)$ and $\Lambda_\alpha(E)$ the corresponding Λ-matrix. Also let $\widetilde{\Lambda}_\alpha(E) = U_E^{1/2} \Lambda_\alpha(E) U_E^{1/2}$ with

$$U_E = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\sqrt{E}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\sqrt{E}} \end{pmatrix}.$$

Explicitly, we have

$$\widetilde{\Lambda}_\alpha(E) = \begin{pmatrix} \dfrac{\mathrm{e}^{\mathrm{i}\sqrt{E}}}{T_\alpha(E)} & -\dfrac{R_\alpha(E)}{T_\alpha(E)} \\ \dfrac{L_\alpha(E)}{T_\alpha(E)} & \dfrac{\mathrm{e}^{-\mathrm{i}\sqrt{E}}}{\overline{T(E)}} \end{pmatrix}.$$

Consider the matrix

$$A(E) = \mathbb{E}\left\{\widetilde{\Lambda}_{\alpha(\omega)}(E)^\dagger \widetilde{\Lambda}_{\alpha(\omega)}(E)\right\} = \int \widetilde{\Lambda}_\alpha(E)^\dagger \widetilde{\Lambda}_\alpha(E)\, \mathrm{d}\kappa(\alpha) \geqslant 0$$

where for brevity we write $\alpha(\omega)$ instead of $\alpha_j(\omega)$ with some $j \in \mathbb{Z}$. Let $\beta_+(E)$ be the largest eigenvalue of $A(E)$ and $\beta_-(E)$ be the smallest. It will turn out below that $\beta_+(E) \geqslant 1$. Set $\widetilde{\gamma}(E) = (\log \beta_+(E))/2 \geqslant 0$.

The first main result of the present paper is:

Theorem 1. *Given the Hamiltonian (1) and the distribution κ for the coupling constant α, for all $E > 0$ the resulting Lyapunov exponent satisfies the upper bound*

$$\gamma(E) \leqslant \widetilde{\gamma}(E). \tag{5}$$

In particular, $\gamma(E)$ decays at least like $1/\sqrt{E}$ at infinity.

Proof. Let $\Lambda^{(n)}(E;\omega)$ denote the Λ-matrix for the pair $(H^{(n)}(\omega), H_0)$, which by the factorization property can be represented in the form

$$\Lambda^{(n)}(E;\omega) = U_E^{-n-1/2} \prod_{j=-n}^{n} \widetilde{\Lambda}_{\alpha_j(\omega)}(E) \cdot U_E^{-n-1/2}. \tag{6}$$

In fact, this factorization property is a consequence of the multiplicativity property of the fundamental matrix (see [12] for a proof and for references to earlier work). A brief calculation gives

$$|T^{(n)}(E;\omega)|^{-2} = \tfrac{1}{4} \operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right) + \tfrac{1}{2}. \tag{7}$$

With $\mathbb{E}$ denoting the expectation with respect to the measure $\mathbb{P}$, by Jensen's inequality and (7) we therefore have the estimate

$$\mathrm{e}^{-2\mathbb{E}\{\log|T^{(n)}(E;\omega)|\}} \leqslant \mathbb{E}\left\{|T^{(n)}(E;\omega)|^{-2}\right\} = \tfrac{1}{4}\mathbb{E}\left\{\operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right)\right\} + \tfrac{1}{2}. \tag{8}$$

From the factorization property (6) it follows that

$$\operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right) = \operatorname{tr}\left(\prod_{j=n}^{-n} \widetilde{\Lambda}_{\alpha_j(\omega)}(E)^\dagger \prod_{j=-n}^{n} \widetilde{\Lambda}_{\alpha_j(\omega)}(E)\right). \tag{9}$$

We will now make use of the fact that the $\alpha_k(\omega)$ are IID random variables. For this purpose define the 2×2 matrices $A_j(E) \geqslant 0$ recursively by $A_0 = I$ and

$$A_j(E) = \int \widetilde{\Lambda}_\alpha(E)^\dagger A_{j-1}(E) \widetilde{\Lambda}_\alpha(E)\, \mathrm{d}\kappa(\alpha) \tag{10}$$

such that, in particular, $A(E) = A_1(E)$. Now it is easy to see that

$$\mathbb{E}\left\{\operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right)\right\} = \operatorname{tr}\left(\mathbb{E}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right)\right) = A_{2n+1}(E). \tag{11}$$

We now use the fact that the operator inequality $0 \leqslant A \leqslant A'$ implies $0 \leqslant \operatorname{tr} A \leqslant \operatorname{tr} A'$ and $B^\dagger A B \leqslant B^\dagger A' B$ for all B. In particular, we have $A(E) \leqslant \beta_+(E)\, I$ from which we obtain the recursive estimates $A_j(E) \leqslant \beta_+(E) A_{j-1}(E) \leqslant \cdots \leqslant \beta_+(E)^j\, I$ and hence

$$\mathbb{E}\left\{\operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right)\right\} \leqslant 2\beta_+(E)^{2n+1}. \tag{12}$$

We note that with the same arguments one proves the lower bound

$$2\beta_-(E)^{2n+1} \leqslant \mathbb{E}\left(\operatorname{tr}\left(\Lambda^{(n)}(E;\omega)^\dagger \Lambda^{(n)}(E;\omega)\right)\right).$$

The relation (3), the estimate (12) combined with (8) and Fatou's lemma now imply

$$\gamma(E) \leqslant \frac{1}{2} \lim_{n\to\infty} \frac{\log \mathbb{E}\left\{|T^{(n)}(E;\omega)|^{-2}\right\}}{2n+1}$$
$$\leqslant \frac{1}{2} \lim_{n\to\infty} \frac{\log\left(\beta_+(E)^{2n+1}/2 + \frac{1}{2}\right)}{2n+1} = \frac{1}{2}\log\beta_+(E)$$

which proves the claim (5).

To establish the final claim of the theorem we recall the following well known estimates (see, e.g., [6, 8]):

$$|T_\alpha(E) - 1| + |R_\alpha(E)| \leqslant C \frac{1}{\sqrt{E}} \tag{13}$$

valid for all large $E > 0$ uniformly for all α in the (compact) support of κ for fixed f. Using the estimate (13) in (14) gives the estimate $\beta_+(E) \leqslant 1 + C/\sqrt{E}$ for all large E. Since $\widetilde{\gamma}(E) = (\log \beta_+(E))/2$, this concludes the proof of the theorem. □

Since $\gamma(E) \geqslant 0$, we obviously have the inequality $\beta_+(E) \geqslant 1$ for almost all E. We will now give a direct independent proof of this fact and simultaneously obtain an expression for $\beta_+(E)$. The matrix $A(E)$ may be written in the form

$$A(E) = \begin{pmatrix} a(E) & b(E) \\ \overline{b(E)} & a(E) \end{pmatrix}$$

with

$$a(E) = \int \left(\frac{2}{|T_\alpha(E)|^2} - 1 \right) \mathrm{d}\kappa(\alpha) \tag{14}$$

$$b(E) = -\mathrm{e}^{\mathrm{i}\sqrt{E}} \int \frac{R_\alpha(E)}{T_\alpha(E)^2} \, \mathrm{d}\kappa(\alpha). \tag{15}$$

This gives the two eigenvalues of $A(E)$ in the form

$$\beta_\pm(E) = a(E) \pm |b(E)|. \tag{16}$$

Obviously, $a(E) \geqslant 1$ and hence $\beta_+(E) \geqslant 1$. In fact, $a(E) = 1$ is possible if and only if $R_\alpha(E) = 0$ for almost all α in the support of κ. Then also $b(E) = 0$ and $\beta_+(E) = 1$. Actually, (if $\operatorname{supp} \kappa$ has at least one non-isolated point) we do not believe there are non-trivial f and E for which this holds, but in any case for such Es the Lyapunov exponent vanishes as is easily verified (see also [12]), so this is a trivial confirmation of estimate (5) in this case. In the remaining case we trivially have $\beta_+(E) > 1$.

As an example we consider the random Kronig–Penney model which is formally obtained from $H(\omega)$ by replacing f with the Dirac δ-function at the origin. Then we have (correcting for a misprint on p 232 of [12])

$$T_\alpha(E) = \left(1 + \mathrm{i} \frac{\alpha}{2\sqrt{E}} \right)^{-1} \tag{17}$$

$$R_\alpha(E) = -\mathrm{i} \frac{\alpha}{2\sqrt{E}} \left(1 + \mathrm{i} \frac{\alpha}{2\sqrt{E}} \right)^{-1} \tag{18}$$

and our method still applies. This gives

$$a(E) = 1 + \frac{\langle \alpha^2 \rangle}{4E} \tag{19}$$

$$b(E) = \mathrm{i} \frac{\langle \alpha \rangle}{2\sqrt{E}} - \frac{\langle \alpha^2 \rangle}{4E}. \tag{20}$$

Here for brevity by $\langle \ \rangle$ we denote the mean with respect to the probability measure $\mathbb{P}$ such that

$$\langle \alpha \rangle = \mathbb{E}\{\alpha(\omega)\} = \int \alpha \, \mathrm{d}\kappa(\alpha) \qquad \langle \alpha^2 \rangle = \mathbb{E}\{\alpha(\omega)^2\} = \int \alpha^2 \, \mathrm{d}\kappa(\alpha).$$

In particular, equation (19) gives

$$\beta_+(E) = 1 + \frac{\langle\alpha^2\rangle}{4E} + \frac{1}{2\sqrt{E}}\left(\frac{\langle\alpha^2\rangle}{4E} + \langle\alpha\rangle^2\right)^{1/2}. \tag{21}$$

So also in this case $\gamma(E)$ decays at least like $1/\sqrt{E}$ as $E \to \infty$ and at least like $1/E$ if the mean $\langle\alpha\rangle$ of α vanishes, i.e. if on average the coupling constant is zero.

3. The integrated density of states

We denote by $\xi_\alpha(E)$ the spectral shift function for the pair $(H_0 + \alpha f, H_0)$. The second main result of this paper is given by:

Theorem 2. *For all $E > 0$ the spectral shift density $\xi(E)$ for the operator (1) satisfies the following two-sided bound:*

$$\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - r(E) \leqslant \xi(E) \leqslant \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + r(E) \tag{22}$$

where

$$r(E) = \min\left\{\frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\}.$$

In particular, $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\}$ and $r(E)$ decays at least like $1/\sqrt{E}$ at infinity.

Remarks.

(a) One can easily prove the following estimate:

$$\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - 1 \leqslant \xi(E) \leqslant \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + 1$$

which is valid for all $E \in \mathbb{R}$.

(b) By the monotonicity of the spectral shift function with respect to perturbation, $\xi(E) \geqslant 0$ if $\operatorname{supp}\kappa \subset \mathbb{R}_+$ and $\xi(E) \leqslant 0$ if $\operatorname{supp}\kappa \subset \mathbb{R}_-$ for almost all $E > 0$.

(c) For large $E > 0$ by (13)

$$r(E) = \min\left\{\frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)}{1 - |R_{\alpha(\omega)}(E)|}\right\}\right\} = \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)}{1 - |R_{\alpha(\omega)}(E)|}\right\} \leqslant \frac{C}{\sqrt{E}}.$$

(d) In [12] we proved the relation $\xi(E) = N_0(E) - N(E) = \sqrt{E}/\pi - N(E)$, where $N_0(E)$ is the integrated density of states for the free operator H_0. Theorem 2 then gives the following two-sided bound for the integrated density of states:

$$\frac{\sqrt{E}}{\pi} - \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} - r(E) \leqslant N(E) \leqslant \frac{\sqrt{E}}{\pi} - \mathbb{E}\{\xi_{\alpha(\omega)}(E)\} + r(E) \qquad E > 0. \tag{23}$$

There are some other upper bounds on the integrated density of states. A well known result is a one-sided bound due to Kirsch and Martinelli [10, corollary 3.1],

$$N(E) \leqslant \frac{C}{\sqrt{\eta}}\,\mathbb{E}\left\{\int_{-1/2}^{1/2}(E + \eta - V_\omega(x))_+\,\mathrm{d}x\right\}$$

for any $\eta > 0$ and all $E \in \mathbb{R}$. This bound, however, does not reproduce the correct asymptotic behaviour of $N(E)$ in the large-energy limit.

(e) The bounds (5) and (22) are of interest in the context of the Thouless formula (see, e.g., [20])

$$\gamma(E) - \gamma_0(E) = -\int_{\mathbb{R}} \log|E - E'|\,\mathrm{d}\xi(E') \qquad E \in \mathbb{R} \tag{24}$$

where $\gamma_0(E) = [\max(0, -E)]^{1/2}$ is the Lyapunov exponent for H_0. The Thouless formula in the form (24) can be viewed as a subtracted dispersion relation (see, e.g., [12]).

Proof. In [12] we proved (see theorem 3.3 there and its proof) that for any two potentials V_1 and V_2 with (compact) disjoint supports one has

$$\xi(E; H_0 + V_1 + V_2, H_0) = \xi(E; H_0 + V_1, H_0) + \xi(E; H_0 + V_2, H_0) + \xi_{12}(E)$$

with

$$\xi_{12}(E) = -\frac{1}{2\pi \mathrm{i}} \log \frac{1 - R_1(E)\, L_2(E)}{1 - \overline{R_1(E)}\,\overline{L_2(E)}}$$

where $R_k(E)$ and $L_k(E)$ are the right and left reflection coefficients for the Schrödinger equation with the potential V_k, $k = 1, 2$. Actually, theorem 3.3 in [12] states that $|\xi_{12}(E)| \leqslant \frac{1}{2}$ for all $E \geqslant 0$. Now we improve on this estimate. As in [12] we set

$$L_k(E) = a_k(E)\, \mathrm{e}^{\mathrm{i}\delta_k^{(L)}} \qquad R_k(E) = a_k(E)\, \mathrm{e}^{\mathrm{i}\delta_k^{(R)}} \qquad k = 1, 2$$

with $0 \leqslant a_k(E) \leqslant 1$. Moreover, $a_k(E) = 1$ only when $T_k(E) = 0$, which we recall can happen only if $E = 0$. Therefore,

$$\log \frac{1 - R_1(E)\, L_2(E)}{1 - \overline{R_1(E)}\,\overline{L_2(E)}} = \log \frac{1 - a_1(E)a_2(E)\, \mathrm{e}^{\mathrm{i}(\delta_1^{(R)} + \delta_2^{(L)})}}{1 - a_1(E)a_2(E)\, \mathrm{e}^{-\mathrm{i}(\delta_1^{(R)} + \delta_2^{(L)})}}$$

$$= -2\mathrm{i} \arctan \frac{a_1(E)a_2(E) \sin(\delta_1^{(R)} + \delta_2^{(L)})}{1 - a_1(E)a_2(E) \cos(\delta_1^{(R)} + \delta_2^{(L)})}.$$

By means of the inequality $|\arctan x| \leqslant |x|$ we immediately obtain

$$|\xi_{12}(E)| \leqslant \min\left\{\frac{1}{2}, \frac{1}{\pi} \frac{a_1(E)a_2(E)}{1 - a_1(E)a_2(E)}\right\}. \tag{25}$$

Since $0 \leqslant a_k(E) < 1$ we can replace $a_1(E)a_2(E)(1 - a_1(E)a_2(E))^{-1}$ either by $a_1(E)(1 - a_1(E))^{-1}$ or by $a_2(E)(1 - a_2(E))^{-1}$.

Now let us consider the operator $H^{(n)}(\omega)$ for finite n. Applying the inequality (25) we obtain

$$\left|\xi^{(n)}(E; \omega) - \xi_{\alpha_n(\omega)}(E) - \xi_{\alpha_{-n}(\omega)}(E) - \xi^{(n-1)}(E; \omega)\right|$$
$$\leqslant \min\left\{\frac{1}{2}, \frac{1}{\pi} \frac{|R_{\alpha_n(\omega)}(E)|}{1 - |R_{\alpha_n(\omega)}(E)|}\right\} + \min\left\{\frac{1}{2}, \frac{1}{\pi} \frac{|R_{\alpha_{-n}(\omega)}(E)|}{1 - |R_{\alpha_{-n}(\omega)}(E)|}\right\}.$$

Repeating this procedure recursively we obtain

$$\left|\xi^{(n)}(E; \omega) - \sum_{j=-n}^{n} \xi_{\alpha_j(\omega)}(E)\right| \leqslant \sum_{j=-n}^{n} \min\left\{\frac{1}{2}, \frac{1}{\pi} \frac{|R_{\alpha_j(\omega)}(E)|}{1 - |R_{\alpha_j(\omega)}(E)|}\right\}.$$

From the existence of the spectral shift density (2) by the Birkhoff ergodic theorem it follows that

$$\left|\xi(E) - \mathbb{E}\left\{\xi_{\alpha(\omega)}(E)\right\}\right| \leqslant \mathbb{E}\left\{\min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha(\omega)}(E)|}{1-|R_{\alpha(\omega)}(E)|}\right\}\right\}.$$

From the obvious inequality

$$\mathbb{E}\left\{\min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{|R_{\alpha(\omega)}(E)|}{1-|R_{\alpha(\omega)}(E)|}\right\}\right\} \leqslant \min\left\{\frac{1}{2}, \frac{1}{\pi}\mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1-|R_{\alpha(\omega)}(E)|}\right\}\right\}$$

the bound (22) follows.

For large E we have the following asymptotics [6] uniformly in α on compact sets:

$$R_\alpha(E) = \frac{\alpha}{2\mathrm{i}\sqrt{E}}\int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\sqrt{E}t} f(t)\,\mathrm{d}t + \mathrm{O}(E^{-1})$$

$$L_\alpha(E) = \frac{\alpha}{2\mathrm{i}\sqrt{E}}\int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\sqrt{E}t} f(t)\,\mathrm{d}t + \mathrm{O}(E^{-1})$$

such that $R_\alpha(E) = \mathrm{O}(1/\sqrt{E})$ and $L_\alpha(E) = \mathrm{O}(1/\sqrt{E})$. If the single-site potential f has p derivatives in $L^1(\mathbb{R})$ then $L_\alpha(E) = \mathrm{O}(E^{-(p+1)/2})$ and $R_\alpha(E) = \mathrm{O}(E^{-(p+1)/2})$ as $E \to \infty$ [6]. The estimate $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\} = \mathrm{O}(1/\sqrt{E})$ is proposition 3 below. □

As an example we consider again the random Kronig–Penney model. The single-site spectral shift function is given in this case by

$$\xi_\alpha(E) = \frac{1}{\pi}\arctan\left(\frac{\alpha}{2\sqrt{E}}\right) \qquad E > 0.$$

Therefore,

$$\mathbb{E}\left\{\xi_{\alpha(\omega)}(E)\right\} = \frac{1}{\pi}\int_{\mathbb{R}} \arctan\left(\frac{\alpha}{2\sqrt{E}}\right)\mathrm{d}\kappa(\alpha)$$

and thus

$$\left|\mathbb{E}\left\{\xi_{\alpha(\omega)}(E)\right\}\right| \leqslant \frac{\langle|\alpha|\rangle}{2\pi\sqrt{E}}.$$

Using the explicit expression for the reflection amplitude one can easily show that

$$\frac{\langle|\alpha|\rangle}{2\sqrt{E}} + \frac{\langle\alpha^2\rangle}{4E} \leqslant \mathbb{E}\left\{\frac{|R_{\alpha(\omega)}(E)|}{1-|R_{\alpha(\omega)}(E)|}\right\} \leqslant \frac{\langle|\alpha|\rangle}{2\sqrt{E}} + \frac{\langle\alpha^2\rangle}{2E}.$$

We complete this section with an estimate on $\mathbb{E}\{\xi_{\alpha(\omega)}(E)\}$ in the general case. We will prove:

Proposition 3. *There is a constant $c > 0$ independent of E, f, and κ such that for all $E > 0$*

$$\left|\mathbb{E}\left\{\xi_{\alpha(\omega)}(E)\right\}\right| \leqslant \frac{C}{2\sqrt{E}}\mathbb{E}\left\{|\alpha(\omega)|^{1/2}\right\}^2 \int_{-1/2}^{1/2} |f(x)|\,\mathrm{d}x.$$

Let $l^{1/2}(L^1)$ denote the Birman–Solomyak class of measurable functions V for which

$$\|V\|_{l^{1/2}(L^1)} = \left[\sum_{j=-\infty}^{\infty}\left(\int_{j-1/2}^{j+1/2}|V(x)|\,\mathrm{d}x\right)^{1/2}\right]^2 < \infty.$$

The claim of the proposition immediately follows from the following:

Lemma 4. *Let $V \in l^{1/2}(L^1)$. There is a constant c_1 independent of V and E such that*

$$|\xi(E; H_0 + V, H_0)| \leqslant \frac{c_1}{2\sqrt{E}} \|V\|_{l^{1/2}(L^1)}$$

for all $E > 0$.

Proof. As proved in [23] there is a constant $c_2 > 0$ independent of E and V such that

$$|\xi(E; H_0 + V, H_0)| \leqslant C_1 \|V^{1/2} R_0(E + \mathrm{i}0)|V|^{1/2}\|_{\mathcal{J}_1}$$

where $V^{1/2} = \operatorname{sign} V\,|V|^{1/2}$, $R_0(z) = (H_0 - z)^{-1}$ and $\|\cdot\|_{\mathcal{J}_1}$ denotes the trace class norm (see, e.g., [21]). From the proof of proposition 5.6 in [21] it follows that

$$\|V^{1/2} R_0(E + \mathrm{i}0)|V|^{1/2}\|_{\mathcal{J}_1} \leqslant \frac{c_3}{\sqrt{E}} \|V\|_{l^{1/2}(L^1)}$$

for all $E > 0$. □

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